Implementation of Riemann's Explicit Formula for Rational and Gaussian Primes in Sage

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Abstract

The objective of this paper is twofold. First, we study the relation of the zeros of Riemann's zeta function to the distribution of rational primes. We build on computational work done by Riesel and Göhl in 1970 and provide source code for an explicit formula of $\pi(x)$. Second, we study the relation of the zeros of a certain Dirichlet L-function to the distribution of Gaussian primes. In both studies, we make extensive use of Sage to illustrate the effect of the zeros in their respective explicit formulae. In particular, we graphically show that including a greater number of zeros in the explicit formulae leads to stronger approximations for the number of primes less than a given value.

1 Introduction

Let $\pi(x)$ denote the prime counting function, which counts the number of primes less than or equal to x. In other words,

$$\pi(x) = \sum_{p \leq x} 1. \tag{1}$$

This summation requires knowledge about the primality of each value $t \leq x$, which can be computationally difficult for very large values of x. For this reason, an explicit formula would be useful to find (or at least approximate) the number of primes less than or equal to a desired x. Let $\pi_0(x)$ denote such an explicit formula for $\pi(x)$. The primary purpose of this paper is to examine the effect of various terms in Riemann's $\pi_0(x)$ formula, which Riemann published in 1859 [1], and von Mangoldt later proved in 1895 [2]. In particular, we illustrate the significance of the famous Riemann zeta function, which is given by:

$$\zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - p^{-s}}, \qquad \Re(s) > 1, \tag{2}$$

and whose functional equation is given by:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \qquad s \neq 1. \tag{3}$$

We first present computations with von Mangoldt's explicit formula for Chebyshev's ψ function. Following this, we build upon computational work done by Riesel and Göhl in 1970 on Riemann's $\pi_0(x)$. In the final sections, we begin with the L-function associated to the Dirichlet character modulo 4 and work toward an explicit formula for the number of Gaussian primes within a given norm. We then present some computations related to an explicit formula for Gaussian primes. The source code for all illustrations has been made available on the author's GitHub repository [3].

2 $\psi_0(\mathbf{x})$ Introduction (Rational Primes)

In the course of his proof of $\pi_0(x)$, von Mangoldt obtained a version of Riemann's explicit formula for Chebyshev's ψ function, which can be expressed with the von Mangoldt function Λ as follows:

$$\psi(x) := \sum_{p^n \leq x} \log(p) = \sum_{k \leq x} \Lambda(k). \tag{4}$$

Beginning with the Riemann zeta function and applying tools from complex analysis [4], von Mangoldt obtained the following version of the explicit formula for $\psi(x)$:

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}, \qquad x > 1, \tag{5}$$

where the sum is over all zeros ρ of the Riemann zeta function. The explicit formula (5) tells us that count of the logarithms of prime powers (scaled appropriately) up to x is approximately x shifted by a constant, plus some oscillating error term.

3 $\psi_0(\mathbf{x})$ Computations (Rational Primes)

Starting from equation (5), we wish to obtain an expression which we can easily write in code for computational purposes. To do this, we first assume that the Riemann hypothesis holds - that is, all non-trivial zeros of $\zeta(s)$ have real part equal to 1/2. We hold this assumption for the remainder of the paper. We can now separate the sum over all zeros into a sum over non-trivial zeros and a sum over trivial zeros:

$$\psi_0(x) = x - \sum_{\gamma} \left(\frac{x^{\frac{1}{2}+i\gamma}}{\frac{1}{2}+i\gamma} + \frac{x^{\frac{1}{2}-i\gamma}}{\frac{1}{2}-i\gamma} \right) - \sum_{m=1}^{\infty} \frac{x^{-2m}}{-2m} - \frac{\zeta'(0)}{\zeta(0)},$$

where γ denotes the positive imaginary parts of the non-trivial zeros in increasing absolute value. In what follows, we commonly refer to γ as the "non-trivial zeros" for brevity. The sum over trivial zeros is the Taylor series expansion for $\frac{1}{2}\log(1 - x^{-2})$, and the term $\frac{\zeta'(0)}{\zeta(0)}$

evaluates to $\log(2\pi)$ [4]. Thus, we have:

$$\psi_0(x) = x - \sum_\gamma \left(\frac{x^{\frac{1}{2}+i\gamma}}{\frac{1}{2}+i\gamma} + \frac{x^{\frac{1}{2}-i\gamma}}{\frac{1}{2}-i\gamma} \right) - \frac{1}{2}\log(1-x^{-2}) - \log(2\pi).$$

If we move the constant $\log(2\pi)$ to the left hand side, we obtain an expression that oscillates around the line $f(x) = x$, with oscillations determined by the location of the zeros of the zeta function. This expression can be handled well in Sage. However, we can obtain a more aesthetic expression through some algebraic manipulations. If we multiply expressions involving the non-trivial zeros by their complex conjugates and expand exponentials with Euler's formula, we see that the summation yields no imaginary part (as expected). In the end, we obtain:

$$\psi_0(x) + \log(2\pi) = x - 4\sqrt{x}\sum_\gamma \frac{\cos(\gamma\log(x)) + 2\gamma\sin(\gamma\log(x))}{1+4\gamma^2} - \frac{1}{2}\log(1-x^{-2}) \tag{6}$$

In the following illustrations, we plot $\psi(x)$ as defined in equation (4), but with the constant $\log(2\pi)$ shifting the step function at $x = 1$ (since the right hand side of (6) is valid for $x > 1$). This allows us to clearly see the effect of including increasingly more zeros of the zeta function.

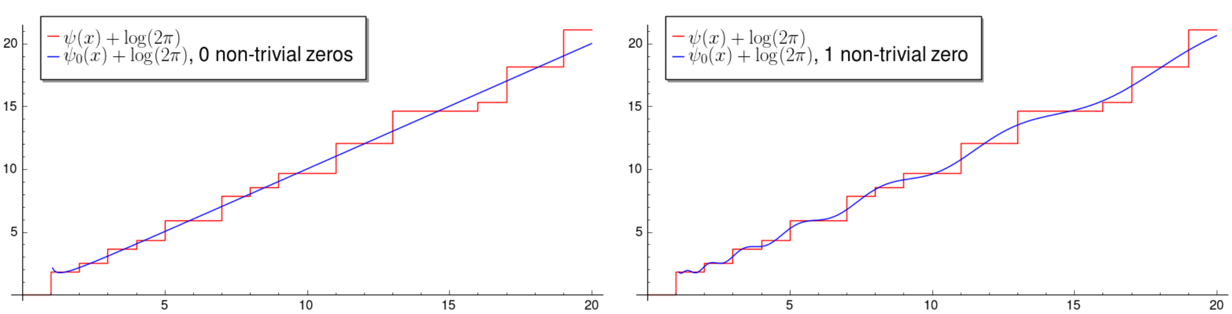

Figure 1: **Left**: $\psi_0(x)$ with 0 non-trivial zeros of the zeta function.
Right: $\psi_0(x)$ with the first non-trivial zero of the zeta function.

We note the upward concavity of $\psi_0(x)$ with 0 non-trivial zeros for small values of x. This is due to the term involving the trivial zeros. As x increases, $\frac{1}{2}\log(1-x^{-2})$ quickly tends to $\frac{1}{2}\log(1) = 0$, leaving only the linear term and the sum over non-trivial zeros at large values of x. On the right side of figure 1, we see that the first non-trivial zero (i.e. first γ) contributes to an oscillation around the line $f(x) = x$.

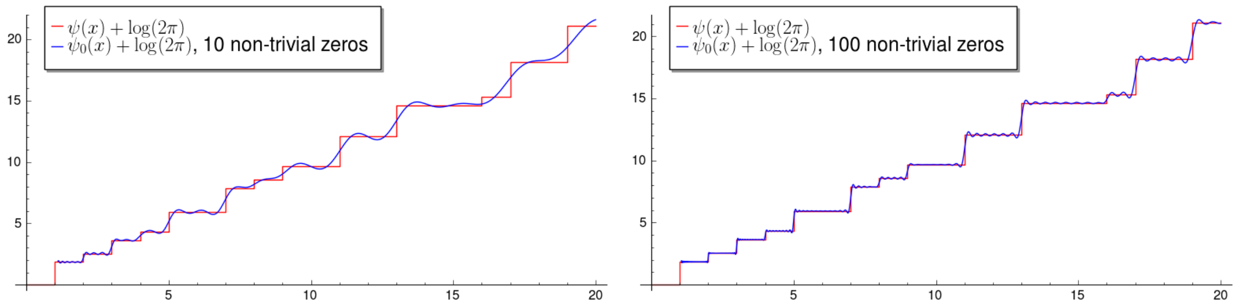

Figure 2: **Left**: $\psi_0(x)$ with the first 10 non-trivial zeros of the zeta function.
Right: $\psi_0(x)$ with the first 100 non-trivial zeros of the zeta function.

As we include increasingly more non-trivial zeros, we see that $\psi_0(x)$ better resembles $\psi(x)$. To further illustrate the effect of the trivial zeros, we perform the same computation as in the right side of figure 2, but without the $\frac{1}{2}\log(1 - x^{-2})$ term.

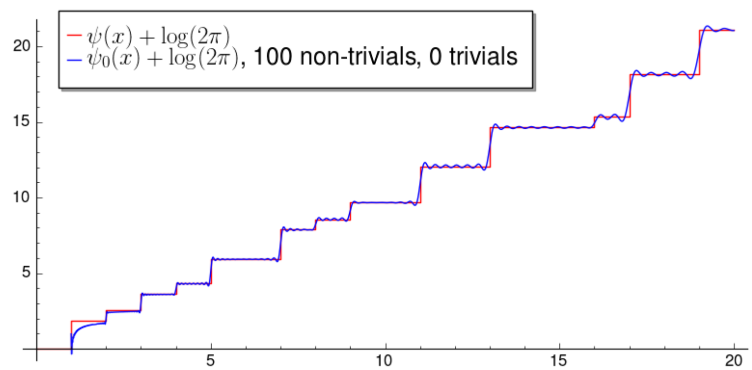

Figure 3: $\psi_0(x)$ with the first 100 non-trivial zeros, but 0 trivial zeros.

Removing the contribution from trivial zeros leads to some inaccuracy for small values of x. However, figure 3 is nearly indistinguishable from the right side of figure 2 at large x.

From our offset $\psi_0(x)$ in equation (6), we can make some interesting approximations. Using the fact that the smallest value for γ is $14.1347\ldots$ [5], together with the fact that $|\cos(\gamma \log(x))| \leq 1$, we make the approximations:

$$\left|\frac{\cos(\gamma \log(x))}{1 + 4\gamma^2}\right| \leq \frac{1}{800.1618\ldots} \approx 0, \qquad \text{and} \qquad 1 + 4\gamma^2 \approx 4\gamma^2.$$

With these approximations, we have:

$$\psi(x) + \log(2\pi) \approx x - 2\sqrt{x}\sum_{\gamma} \frac{\sin(\gamma \log(x))}{\gamma} - \frac{1}{2}\log(1 - x^{-2}). \tag{7}$$

We now see that the oscillation term resembles a Fourier sine series periodic with respect to $\log(x)$, and with amplitudes and frequencies controlled by the locations of the non-trivial zeros on the critical line $\Re(s) = 1/2$.

4 $\pi_0(\mathbf{x})$ Introduction (Rational Primes)

In going from equation (5) to $\pi_0(x)$, we follow von Mangoldt and make use of Riemann's prime power counting function, given by:

$$\Pi(x) := \sum_{p^n \leq x} \frac{1}{n} = \sum_{k \leq x} \frac{\Lambda(k)}{\log(k)}. \tag{8}$$

Using Abel's summation formula, we can write this as:

$$\sum_{k \leq x} \Lambda(k) \left(\int_2^x \frac{dt}{t \log^2(t)} + \frac{1}{\log(x)} \right).$$

But by equation (4), and then using integration by parts, we have:

$$\begin{aligned} \Pi(x) &= \int_2^x \frac{\psi(t)}{t \log^2(t)} dt + \frac{\psi(x)}{\log(x)} \\ &= \int_2^x \frac{\psi'(t)}{\log(t)} dt. \end{aligned}$$

Recalling the definition of the offset logarithmic integral:

$$\mathrm{li}(x) := \int_2^x \frac{dt}{\log(t)}, \tag{9}$$

we now have, by way of equation (5), an explicit formula for Riemann's prime power counting function:

$$\Pi_0(x) = \mathrm{li}(x) + \sum_{\rho} \mathrm{li}(x^{\rho}), \qquad x > 1. \tag{10}$$

Since

$$\pi(x) := \sum_{p \leq x} 1, \qquad \text{and} \qquad \Pi(x) := \sum_{p^n \leq x} \frac{1}{n},$$

we have:

$$\Pi(x) = \sum_{n>0} \frac{\pi(x^{1/n})}{n}.$$

By applying möbius inversion, we can obtain an explicit formula for $\pi(x)$:

$$\begin{aligned}
\pi_0(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \Pi(x^{1/n}) \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left[\mathrm{li}(x^{1/n}) - \sum_{\rho} \mathrm{li}(x^{\rho/n}) \right] \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{1/n}) - \sum_{\rho} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{\rho/n}),
\end{aligned} \tag{11}$$

where $\mu(n)$ denotes the möbius function. We can write $\pi_0(x)$ more concisely as:

$$\pi_0(x) = \mathrm{R}(x) - \sum_{\rho} \mathrm{R}(x^{\rho}), \tag{12}$$

where the sum is over all zeros ρ of the Riemann zeta function, and $\mathrm{R}(x)$ is given by:

$$\mathrm{R}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{1/n}).$$

Analogous to the linear term in equation (5), we see that $\pi_0(x)$ has $\mathrm{R}(x)$ as a main term, which gives a smooth approximation to the number of primes less than or equal to x. The second term in equation (12) can be considered an oscillating correction term involving the zeros of the zeta function.

5 $\pi_0(\mathbf{x})$ Computations (Rational Primes)

As in section 3 for $\psi_0(x)$, we now wish to obtain an expression for $\pi_0(x)$ which we can easily write in code for computational purposes. Starting with equation (12), we expand the sum over all zeros into a sum over non-trivial zeros and a sum over trivial zeros under the assumption of the Riemann hypothesis:

$$\begin{aligned}
\pi_0(x) &= \mathrm{R}(x) - \sum_{\gamma} \left(\mathrm{R}(x^{1/2+i\gamma}) + \mathrm{R}(x^{1/2-i\gamma})\right) - \sum_{m=1}^{\infty} \mathrm{R}(x^{-2m}) \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{1/n}) - \sum_{\gamma} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\mathrm{li}(x^{\frac{1/2+i\gamma}{n}}) + \mathrm{li}(x^{\frac{1/2-i\gamma}{n}})\right) - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{\frac{-2m}{n}}) \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{1/n}) - \sum_{\gamma} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} 2\Re \left(\mathrm{li}(x^{\frac{1/2+i\gamma}{n}})\right) - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{\frac{-2m}{n}}). \qquad (13)
\end{aligned}$$

The main problem with implementing (13) in Sage comes from the term involving the non-trivial zeros. Sage has difficulty handling this expression due to the behavior of the logarithmic integral at its branch points when given complex arguments. From here, there are two options. The first is to use an approximation, which Hans Riesel and Gunnar Göhl used in 1970 while performing computations with the first 29 pairs of non-trivial zeros [6]. We highlight a potential problem that may arise if very high precision is desired. Alternatively, we can express the logarithmic integral in terms of the exponential integral. This yields a more accurate result for large x, but comes at the cost of additional computation time. We then show the effect of the zeta function's non-trivial zeros in the explicit formula. Finally, we build on Riesel and Göhl's approximation to show a rather aesthetic result that emphasizes the Fourier-like traits of Riemann's explicit formula.

From the prime number theorem [7], we have that:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1, \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\pi(x)}{\mathrm{li}(x)} = 1,$$

where $\mathrm{li}(x)$ is the offset logarithmic integral in equation (9). In their computations, Riesel and Göhl use the approximation $\mathrm{li}(x) \sim x/\log(x)$ [6]. Thus, for non-trivial zeros ρ of $\zeta(s)$:

$$\mathrm{li}(x^\rho) \sim \frac{x^\rho}{\rho \log(x)}.$$

With this approximation, the sum over non-trivial zeros in equation (13) becomes:

$$\sum_\gamma \sum_{n=1}^{\infty} \frac{\mu(n)}{n} 2\Re\left(\mathrm{li}(x^{\frac{1/2+i\gamma}{n}})\right) \approx \sum_\gamma \sum_{n=1}^{\infty} \frac{\mu(n)}{n} 2\Re\left(\frac{x^{\frac{1/2+i\gamma}{n}}}{\left(\frac{1/2+i\gamma}{n}\right)\log(x)}\right).$$

Then operating on the summand, we have:

$$\begin{aligned}
2\Re\left(\frac{x^{\frac{1/2+i\gamma}{n}}}{\left(\frac{1/2+i\gamma}{n}\right)\log(x)}\right) &= 2\Re\left(\frac{(\sqrt{x})^{1/n}}{\log(x)} \frac{e^{\frac{i\gamma}{n}\log(x)}}{\frac{1}{2n}+\frac{i\gamma}{n}}\right) \\
&= \frac{2(\sqrt{x})^{1/n}}{\log(x)} \Re\left(\frac{\cos(\frac{\gamma}{n}\log(x)) + i\sin(\frac{\gamma}{n}\log(x))}{\frac{1}{2n}+\frac{i\gamma}{n}}\right) \\
&= \frac{4n(\sqrt{x})^{1/n}}{\log(x)} \left(\frac{\cos(\frac{\gamma}{n}\log(x)) + 2\gamma\sin(\frac{\gamma}{n}\log(x))}{1+4\gamma^2}\right).
\end{aligned}$$

Then, with the approximation $\mathrm{li}(x) \sim x/\log(x)$, the explicit formula is of the form:

$$\begin{aligned}
\pi_0(x) \approx \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\mathrm{li}(x^{1/n}) - \sum_\gamma \sum_{n=1}^{\infty} \mu(n) \frac{4(\sqrt{x})^{1/n}}{\log(x)} \left(\frac{\cos(\frac{\gamma}{n}\log(x)) + 2\gamma\sin(\frac{\gamma}{n}\log(x))}{1+4\gamma^2}\right) \\
- \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\mathrm{li}(x^{-2m/n}).
\end{aligned} \tag{14}$$

Alternatively, we can obtain a more precise expression by using the relation $\mathrm{li}(x^\rho) = \mathrm{Ei}(\rho\log(x))$, where Ei is the exponential integral function for complex arguments with positive real part. Applying this relation directly to equation (13), we obtain:

$$\pi_0(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\mathrm{li}(x^{1/n}) - \sum_\gamma \sum_{n=1}^{\infty} \frac{\mu(n)}{n} 2\Re\left(\mathrm{Ei}\left(\frac{\frac{1}{2}+i\gamma}{n}\log(x)\right)\right) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\mathrm{li}(x^{\frac{-2m}{n}}). \tag{15}$$

Both equations (14) and(15) are easily handled in Sage, and the corresponding code can be found in [3]. However, the use of the exponential integral function in (15) results in a significantly longer computation time when compared with equation (14) for equivalent values of $x, n,$ and γ. If one wishes to maximize computational accuracy for a given number of non-trivial zeros, equation (15) should be used. The following plots compare the behavior of $\frac{x^\rho}{\rho \log(x)}$ against $\mathrm{Ei}(\rho \log(x))$ for sample values of x and ρ.

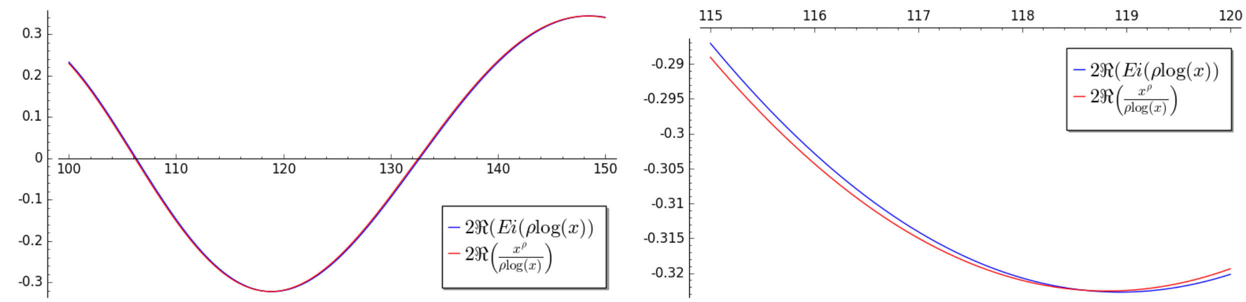

Figure 4: **Left**: $2\Re(\mathrm{Ei}(\rho \log(x)))$ vs. $2\Re\left(\frac{x^\rho}{\rho \log(x)}\right)$ for $\rho = \frac{1}{2} + i \cdot 14.1347\ldots$
Right: Close-up of left for $x \in [115, 120]$.

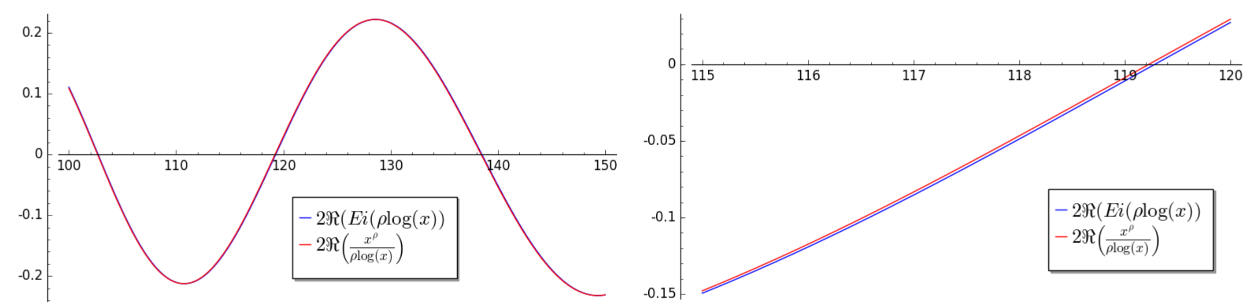

Figure 5: **Left**: $2\Re(\mathrm{Ei}(\rho \log(x)))$ vs. $2\Re\left(\frac{x^\rho}{\rho \log(x)}\right)$ for $\rho = \frac{1}{2} + i \cdot 21.0220\ldots$
Right: Close-up of left for $x \in [115, 120]$.

We now wish to examine the effect of the zeta function's non-trivial zeros in the explicit formula presented in equation (15). As we include increasingly more non-trivial zeros in the correction term, Riemann's explicit formula more closely resembles the prime-counting step function. Following Riesel and Göhl, we always choose our N (the cutoff value for the sum over n) such that $2^{N+1} > x$ [6]. In addition, we include the first 100 trivial zeros in our computations of $\pi_0(x)$. Additional details regarding the implementation of equation (15) can be found in [3].

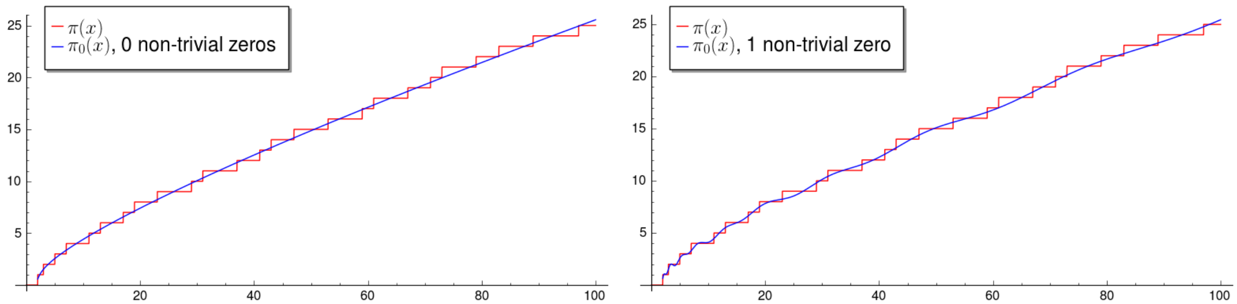

Figure 6: **Left**: $\pi_0(x)$ with 0 non-trivial zeros of the zeta function.
Right: $\pi_0(x)$ with the first non-trivial zero of the zeta function.

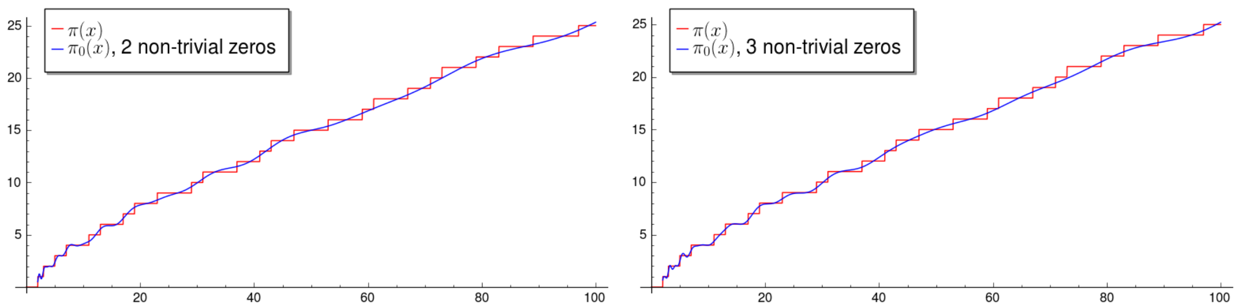

Figure 7: **Left**: $\pi_0(x)$ with the first 2 non-trivial zeros of the zeta function.
Right: $\pi_0(x)$ with the first 3 non-trivial zeros of the zeta function.

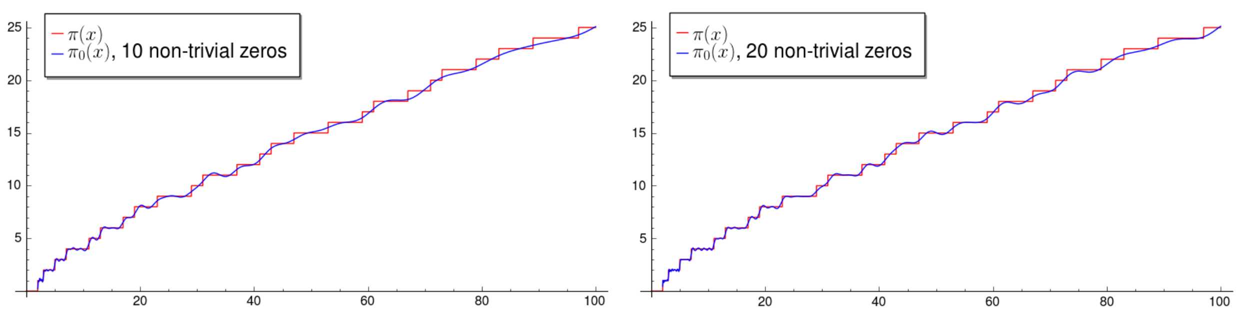

Figure 8: **Left**: $\pi_0(x)$ with the first 10 non-trivial zeros of the zeta function.
Right: $\pi_0(x)$ with the first 20 non-trivial zeros of the zeta function.

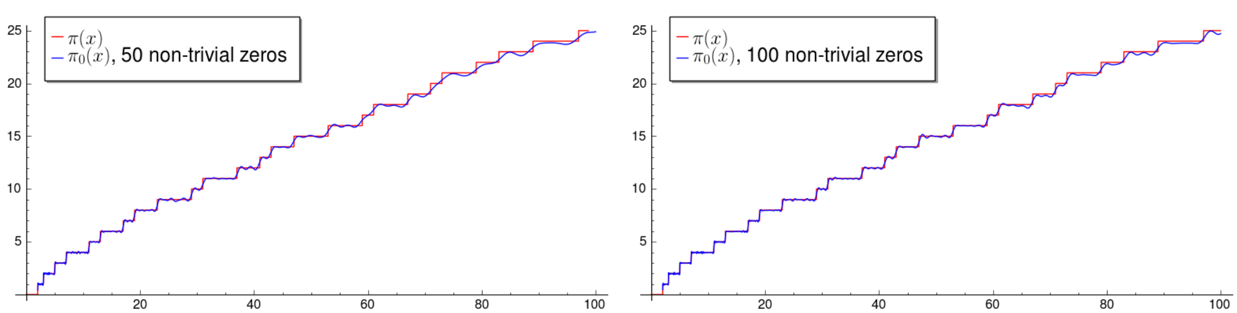

Figure 9: **Left**: $\pi_0(x)$ with the first 50 non-trivial zeros of the zeta function.
Right: $\pi_0(x)$ with the first 100 non-trivial zeros of the zeta function.

From equation (14), we can make an interesting approximation. Using an identical approximation as in equation (7) for $\cos(\frac{\gamma}{n}\log(x))$ and $1+4\gamma^2$, we can further approximate the explicit formula to:

$$\pi_0(x) \approx \mathrm{R}(x) - \sum_{n=1}^{\infty} \mu(n) \frac{(\sqrt{x})^{1/n}}{\log(x)} 2 \sum_{\gamma} \frac{\sin(\frac{\gamma}{n}\log(x))}{\gamma} - \sum_{m=1}^{\infty} \mathrm{R}(x^{-2m}). \tag{16}$$

We note the similarity to equation (7). We have a main term, namely $\mathrm{R}(x)$, with a small correction term involving a sum over trivial zeros, and an oscillating correction term resembling a Fourier sine series periodic in $\log(x)$ whose amplitudes and frequencies are controlled by the location of non-trivial zeros on the critical line $\Re(s) = 1/2$.

6 Gaussian Primes

A *Gaussian integer* is a complex number whose real and imaginary parts are both integers. The Gaussian integers form an integral domain, which we denote with $\mathbb{Z}[i]$. In other words, we have:

$$\mathbb{Z}[i] = \{a + bi \quad | \quad a, b \in \mathbb{Z}\}.$$

The units of the Gaussian integers are $\pm i$ and ± 1 [8]. We say that an element in $\mathbb{Z}[i]$ is a *Gaussian prime* if it is irreducible - that is, if its only divisors are itself and a unit of $\mathbb{Z}[i]$. As with the rational primes, one might inquire about the number of Gaussian primes less than a given value.

The *norm* function takes a Gaussian integer $a + bi$ and maps it to a strictly positive real value. We define the norm of a Gaussian integer as $N(a + bi) = (a + bi)(\overline{a + bi}) = a^2 + b^2$. Thus, if a rational prime p can be written as a sum of squares, it is the norm of two unique elements in $\mathbb{Z}[i]$ closer to the origin. One can show that all primes p congruent to 1 modulo 4 can be written as a sum of squares, and are thus reducible [9]. Geometrically, the norm represents a Gaussian integer's squared distance from the origin. As we will soon see, the norm is an easier measure to use than distance when working with the explicit formula for Gaussian primes.

One can show that an element in $\mathbb{Z}[i]$ is prime if it falls into one of three cases [8]. Let p be a rational prime and let u be a unit of $\mathbb{Z}[i]$. Then Gaussian primes satisfy one of:

- $u(1+i)$ Since $p = 2 = N(1+i)$
- $u(a \pm bi)$ $a^2 + b^2 = p \equiv 1 \pmod 4$
- $u(p)$ $p \equiv 3 \pmod 4$.

If we let x represent the norm, we can obtain a prime-counting function for number of Gaussian primes in any one quadrant within a given norm [9]. We denote this as:

$$\pi_G(x) = 2\pi_{4,1}(x) + \pi_{4,3}(\sqrt{x}) + 1, \tag{17}$$

where $\pi_{4,3}(x)$ and $\pi_{4,1}(x)$ are the respective counting functions for rational primes in $4n+3$ and $4n+1$ arithmetic progressions. As we did with the rational primes, we now wish to obtain an explicit formula for Gaussian primes.

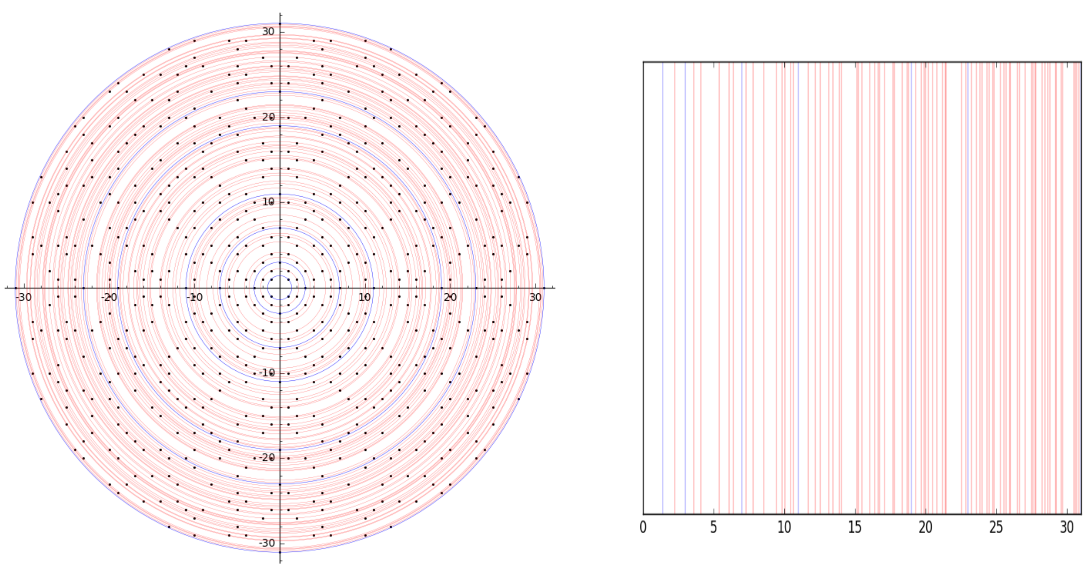

Figure 10: **Left**: A plot of Gaussian primes (black dots) within a norm of 961 (i.e. a distance of 31). Concentric circles have been overlaid as a visual aid. Blue circles correspond to primes congruent to 3 modulo 4 (and those with norm equal to 2). Red circles correspond to Gaussian primes whose norm is a rational prime congruent to 1 modulo 4.
Right: If we imagine walking along a ray extending away from the origin, we add 1 to our count of (unique) Gaussian primes whenever we cross a blue line, and 2 whenever we cross a red line. Scale denotes distance from the origin.

7 $\psi_0(\mathbf{x}, \chi)$ Introduction (Gaussian Primes)

Let χ be the non-principal Dirichlet character modulo 4 (i.e. the character defined by $\chi(0) = 0$, $\chi(1) = 1$, $\chi(2) = 0$, $\chi(3) = -1$). We then have an L-function analogous to the Riemann zeta function given by:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}}, \qquad \Re(s) > 1, \tag{18}$$

and a Chebyshev ψ-type function similar to equation (4) given by:

$$\psi(x, \chi) = \sum_{k \leq x} \chi(k)\Lambda(k). \tag{19}$$

Applying similar tools from complex analysis [10] in the derivation of equation (5), we arrive at an explicit formula for $\psi(x, \chi)$ given by:

$$\psi_0(x, \chi) = -\sum_{\rho_*} \frac{x^{\rho_*}}{\rho_*} - \frac{L'(0, \chi)}{L(0, \chi)}, \tag{20}$$

where ρ_* is a zero of the L-function in equation (18).

8 $\psi_0(\mathbf{x}, \chi)$ Computations (Gaussian Primes)

We now wish to obtain an expression which we can easily write in code. To do this, we assume that the generalized Riemann hypothesis for L-functions holds - that is, all non-trivial zeros have real part equal to 1/2. We can now separate the sum over all zeros ρ_* in equation (20) into a sum over non-trivial zeros[1] and a sum over trivial zeros, which occur at negative odd integers. From equation (20), we have:

[1]The imaginary parts of the non-trivial zeros were obtained using Michael Rubinstein's lcalc library [11].

$$\psi_0(x,\chi) = -\sum_{\gamma_*}\left(\frac{x^{\frac{1}{2}+i\gamma_*}}{\frac{1}{2}+i\gamma_*}+\frac{x^{\frac{1}{2}-i\gamma_*}}{\frac{1}{2}-i\gamma_*}\right) - \sum_{m=1}^{\infty}\frac{x^{-2m+1}}{-2m+1} - \frac{\mathrm{L}'(0,\chi)}{\mathrm{L}(0,\chi)}$$
$$= -4\sqrt{x}\sum_{\gamma_*}\left(\frac{\cos(\gamma_*\log(x))+2\gamma_*\sin(\gamma_*\log(x))}{1+4\gamma_*^2}\right)+\tanh^{-1}\left(\frac{1}{x}\right)-\frac{\mathrm{L}'(0,\chi)}{\mathrm{L}(0,\chi)},$$

where γ_* denotes the positive imaginary parts of the L-function's non-trivial zeros in increasing absolute value. As we did for γ in earlier sections, we commonly refer to γ_* as the L-function's "non-trivial zeros" for brevity. The constant $\frac{\mathrm{L}'(0,\chi)}{\mathrm{L}(0,\chi)}$ was numerically evaluated using Sage to 0.783188740527402 [11].

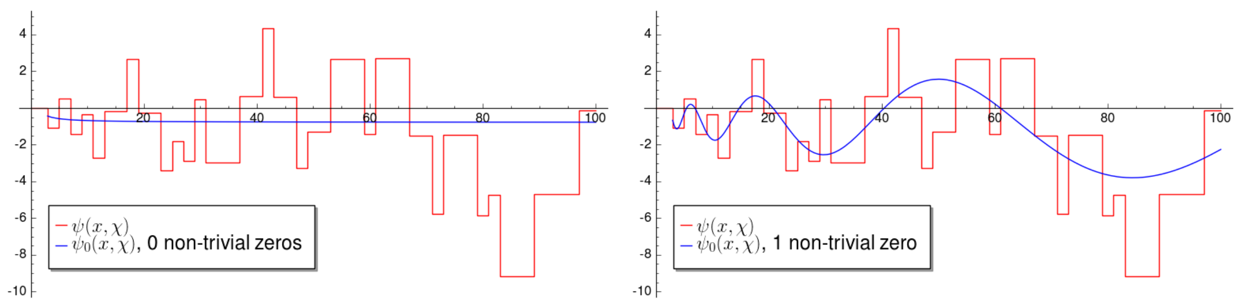

Figure 11: **Left**: $\psi_0(x,\chi)$ with 0 non-trivial zeros of the L-function.
Right: $\psi_0(x,\chi)$ with the first non-trivial zero of the L-function.

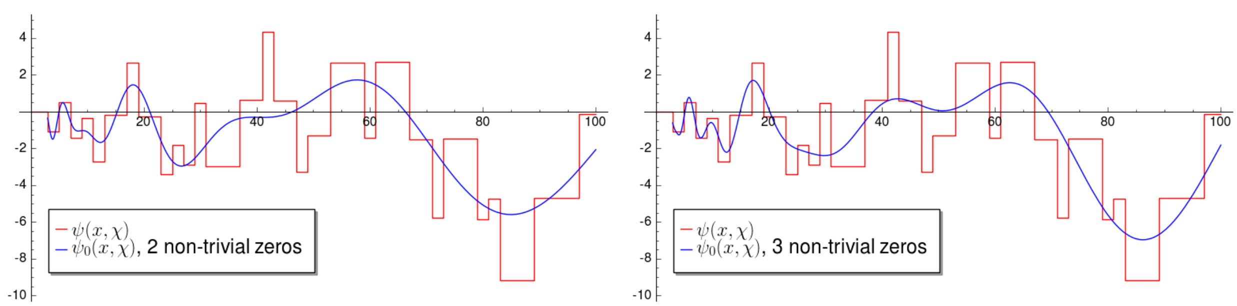

Figure 12: **Left**: $\psi_0(x,\chi)$ with the first 2 non-trivial zeros of the L-function.
Right: $\psi_0(x,\chi)$ with the first 3 non-trivial zeros of the L-function.

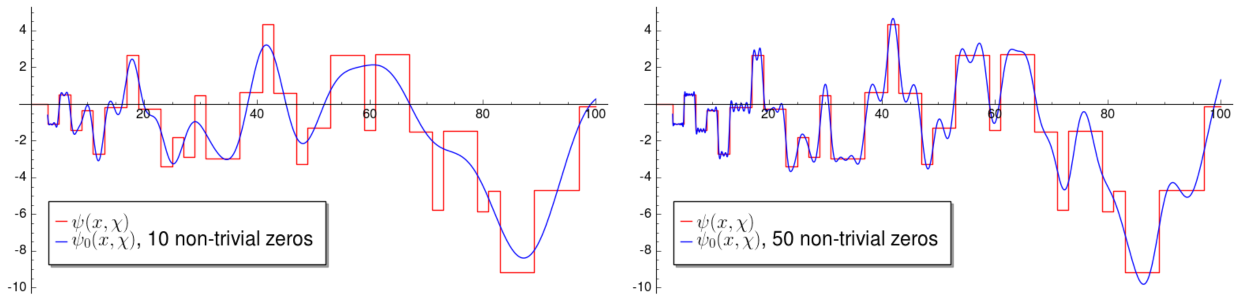

Figure 13: **Left**: $\psi_0(x,\chi)$ with the first 10 non-trivial zeros of the L-function.
Right: $\psi_0(x,\chi)$ with the first 50 non-trivial zeros of the L-function.

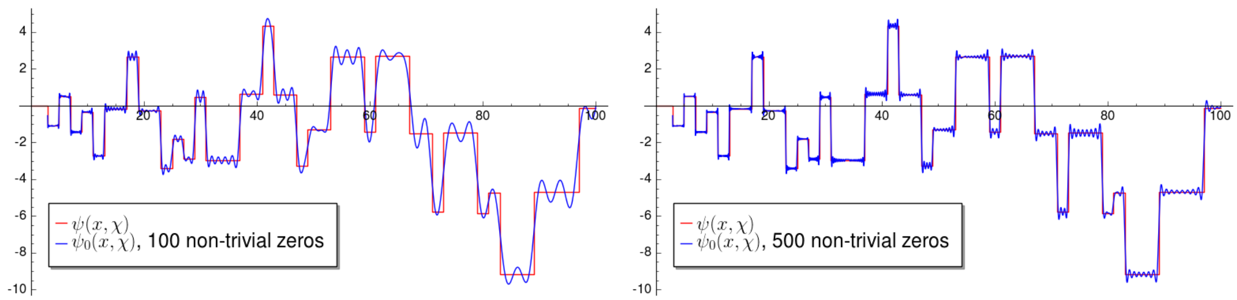

Figure 14: **Left**: $\psi_0(x,\chi)$ with the first 100 non-trivial zeros of the L-function.
Right: $\psi_0(x,\chi)$ with the first 500 non-trivial zeros of the L-function.

9 $\pi_0^G(x)$ Introduction (Gaussian Primes)

We introduce an analogue to the Riemann prime power counting function, given by:

$$\Pi(x,\chi) := \sum_{p^n \le x} \frac{\chi(p^n)}{n} = \sum_{k \le x} \frac{\chi(k)\Lambda(k)}{\log(k)}. \tag{21}$$

Using Abel's summation formula, we can write this as:

$$\sum_{k \le x} \chi(k)\Lambda(k) \left(\int_2^x \frac{dt}{t\log^2(t)} + \frac{1}{\log(x)} \right).$$

But by equation (19), and then using integration by parts, we have:

$$\begin{aligned}\Pi(x,\chi) &= \int_2^x \frac{\psi(t,\chi)}{t\log^2(t)}dt + \frac{\psi(x,\chi)}{\log(x)} \\ &= \int_2^x \frac{\psi'(t,\chi)}{\log(t)}dt.\end{aligned}$$

Recalling the offset logarithmic integral in equation (9), we now have, by way of equation (20), an explicit formula for the prime power counting function in equation (21):

$$\Pi_0(x,\chi) = -\sum_{\rho_*} \mathrm{li}(x^{\rho_*}), \qquad x > 1. \tag{22}$$

Since

$$\pi(x,\chi) := \sum_{p\le x} \chi(p), \qquad \text{and} \qquad \Pi(x,\chi) := \sum_{p^n \le x} \frac{\chi(p^n)}{n},$$

we have:

$$\begin{aligned}\Pi(x,\chi) &= \sum_{n>0} \frac{\pi(x^{1/n},\chi^n)}{n} \\ &= \sum_{n>0} \frac{\pi_{4,1}(x^{1/n}) + (-1)^n \pi_{4,3}(x^{1/n})}{n} \\ &= \sum_{n>0} \frac{\pi_{4,1}(x^{1/n}) - \pi_{4,3}(x^{1/n}) + \pi_{4,3}(x^{1/2n})}{n}.\end{aligned} \tag{23}$$

After applying möbius inversion to equations (22) and (23) we obtain:

$$\pi_{4,1}(x) - \pi_{4,3}(x) + \pi_{4,3}(\sqrt{x}) = -\sum_{\rho_*} \mathrm{R}(x^{\rho_*}).$$

We now have a relation between the zeros of the L-function in equation (18) and counts of primes in arithmetic progressions[2] modulo 4. To obtain a relation to the Gaussian prime-counting function, we use equation (17):

[2] For readers interested in an explicit formula for primes in arithmetic progressions, see [10].

$$\pi_{4,1}(x) + \pi_{4,3}(\sqrt{x}) = \pi_{4,3}(x) - \sum_{\rho_*} \mathrm{R}(x^{\rho_*}),$$

$$2\pi_{4,1}(x) + \pi_{4,3}(\sqrt{x}) = \pi_{4,1}(x) + \pi_{4,3}(x) - \sum_{\rho_*} \mathrm{R}(x^{\rho_*}),$$

$$2\pi_{4,1}(x) + \pi_{4,3}(\sqrt{x}) + 1 = \pi(x) - \sum_{\rho_*} \mathrm{R}(x^{\rho_*}).$$

So our explicit formula for the number of Gaussian primes within a given norm x is:

$$\pi_0^G(x) = \mathrm{R}(x) - \sum_{\rho} \mathrm{R}(x^{\rho}) - \sum_{\rho_*} \mathrm{R}(x^{\rho_*}), \tag{24}$$

where ρ are the zeros of the Riemann zeta function in equation (2), and ρ_* are the zeros of the L-function in equation (18).

10 $\pi_0^G(x)$ Computations (Gaussian Primes)

We now wish to obtain an expression for $\pi_0^G(x)$ which we can easily write in code. Starting with equation (24), we expand the sums over all zeros into sums over non-trivial zeros and sums over trivial zeros under the assumption of the generalized Riemann hypothesis:

$$\begin{aligned}\pi_0^G(x) = \mathrm{R}(x) &- \sum_{\gamma} \left(\mathrm{R}(x^{1/2+i\gamma}) + \mathrm{R}(x^{1/2-i\gamma})\right) - \sum_{m=1}^{\infty} \mathrm{R}(x^{-2m}) \\ &- \sum_{\gamma_*} \left(\mathrm{R}(x^{1/2+i\gamma_*}) + \mathrm{R}(x^{1/2-i\gamma_*})\right) - \sum_{m=1}^{\infty} \mathrm{R}(x^{-2m+1}).\end{aligned} \tag{25}$$

Since we assume the non-trivial zeros of both the zeta function and the L-function lie on critical line $\Re(s) = 1/2$, we can sort these zeros by the size of their imaginary part. Let θ denote the positive imaginary part of a non-trivial zero of either the zeta function or the L-function. In other words, we had $\gamma \in \{14.134..., 21.022..., 25.011..., 30.425..., 32.935..., \ldots\}$, and $\gamma_* \in \{6.021..., 10.244..., 12.988..., 16.343..., 18.292..., \ldots\}$. After combining and sorting these lists, we define θ such that $\theta \in \{6.021..., 10.244..., 12.988..., 14.134..., 16.343..., \ldots\}$.

After combining the trivial zeros, we obtain zeros at all negative integers. Our explicit formula is then:

$$
\begin{aligned}
\pi_0^G(x) &= \mathrm{R}(x) - \sum_{\theta}\left(\mathrm{R}(x^{1/2+i\theta}) + \mathrm{R}(x^{1/2-i\theta})\right) - \sum_{m=1}^{\infty}\mathrm{R}(x^{-m}) \\
&= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathrm{li}(x^{1/n}) - \sum_{\theta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\mathrm{li}(x^{\frac{1/2+i\theta}{n}}) + \mathrm{li}(x^{\frac{1/2-i\theta}{n}})\right) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathrm{li}(x^{\frac{-m}{n}}) \\
&= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathrm{li}(x^{1/n}) - \sum_{\theta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}2\Re\left(\mathrm{li}(x^{\frac{1/2+i\theta}{n}})\right) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathrm{li}(x^{\frac{-m}{n}}) \\
&= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathrm{li}(x^{1/n}) - \sum_{\theta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}2\Re\left(\mathrm{Ei}\left(\frac{\frac{1}{2}+i\theta}{n}\log(x)\right)\right) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathrm{li}(x^{\frac{-m}{n}}).
\end{aligned}
\tag{26}
$$

Equation (26) is almost identical to equation (15). The only differences are the non-trivial zeros occurring more frequently on the line $\Re(s) = 1/2$, and the trivial zeros occurring at all negative integers.

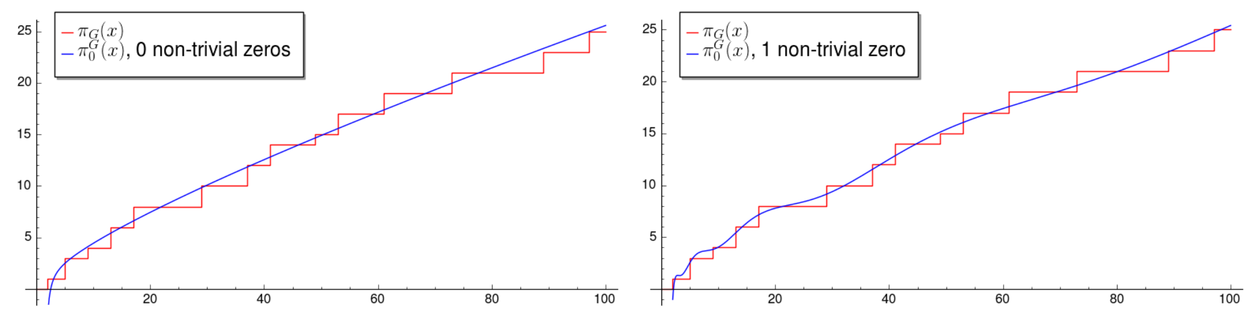

Figure 15: **Left**: $\pi_0^G(x)$ with 0 non-trivial zeros.
Right: $\pi_0^G(x)$ with the first non-trivial zero.

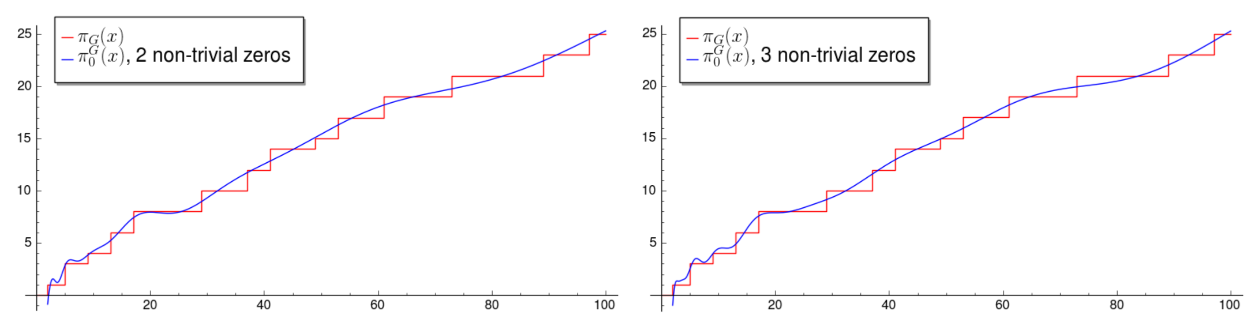

Figure 16: **Left**: $\pi_0^G(x)$ with the first 2 non-trivial zeros.
Right: $\pi_0^G(x)$ with the first 3 non-trivial zeros.

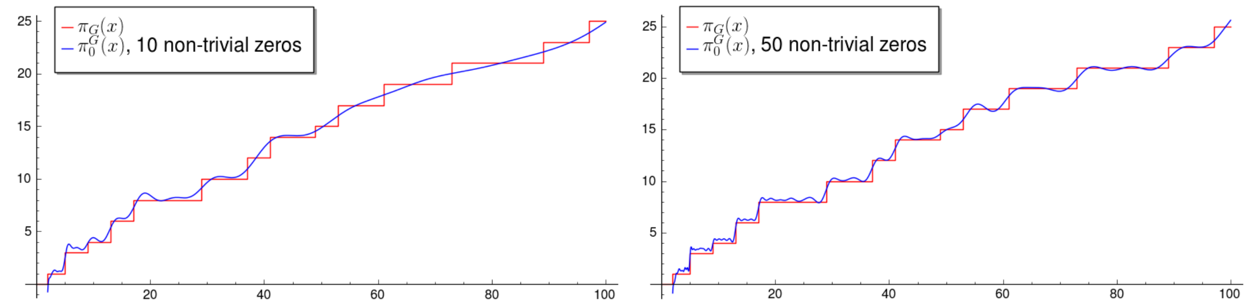

Figure 17: **Left**: $\pi_0^G(x)$ with the first 10 non-trivial zeros.
Right: $\pi_0^G(x)$ with the first 50 non-trivial zeros.

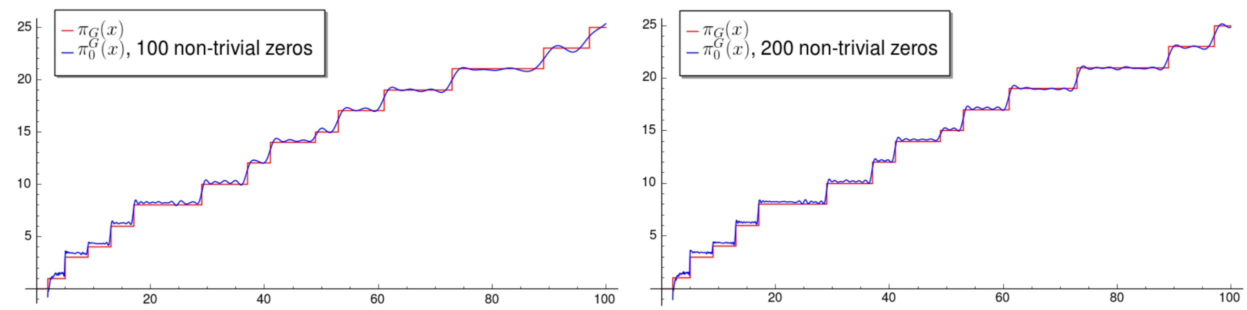

Figure 18: **Left**: $\pi_0^G(x)$ with the first 100 non-trivial zeros.
Right: $\pi_0^G(x)$ with the first 200 non-trivial zeros.

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References

[1] B. Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse," *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin.*, pp. 671–680, Nov. 1859. [Online]. Available: claymath.org/sites/default/files/zeta.pdf 1

[2] H. von Mangoldt, “Zu Riemanns Abhandlung ”Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse”.” *Journal für die reine und angewandte Mathematik*, vol. 114, pp. 255–305, 1895. [Online]. Available: http://eudml.org/doc/148949 1

[3] D. Hutama. GitHub Repository - Riemann’s Explicit Formula for Primes. [Online]. Available: https://github.com/danielhutama 1, 8, 9

[4] H. Davenport, *Multiplicative Number Theory*, 2nd ed. Springer-Verlag Berlin Heidelberg GmbH, pp. 104-110. 2, 3

[5] LMFDB. Zeros of zeta(s). [Online]. Available: http://www.lmfdb.org/zeros/zeta/ 4

[6] H. Riesel and G. Göhl, “Some Calculations Related to Riemann’s Prime Number Formula,” *Mathematics of Computation*, vol. 24, no. 112, pp. 969–983, Oct. 1970. 7, 8, 9

[7] E. W. Weisstein. Prime Number Theorem. [Online]. Available: http://mathworld.wolfram.com/PrimeNumberTheorem.html 7

[8] E. W. Weisstein. Gaussian Integer. [Online]. Available: http://mathworld.wolfram.com/GaussianInteger.html 11, 12

[9] D. Hutama, “Modeling Chebyshev’s Bais in the Gaussian Primes as a Random Walk,” *Comptes rendus des stages de recherche de l’ISM*, July 2016. [Online]. Available: http://ism.uqam.ca/~ism/pdf/Hutama.pdf 11, 12

[10] R. Manzoni, “Riemann’s Explicit Formulas,” *StackExchange - Mathematics*, Jan. 2013. [Online]. Available: https://math.stackexchange.com/questions/269997/two-representations-of-the-prime-counting-function/282848 13, 16

[11] M. Rubinstein, “Rubinstein’s lcalc Library,” July 2009. [Online]. Available: http://doc.sagemath.org/html/en/reference/libs/sage/libs/lcalc/lcalc_Lfunction.html 13, 14

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